# Gamma Function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx$$

$\boxed{1}$ $\Gamma(1) = 1$ $\qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\boxed{2}$ $\Gamma(n) = (n-1)!$

$\boxed{3}$ $\Gamma(x) \cdot \Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$

$\boxed{4}$ $\Gamma(\alpha+1) = \alpha \, \Gamma(\alpha)$

$\boxed{\text{Ex}}$ show that $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

$\boxed{\text{Sol}}$

$$\Gamma(\alpha+1) = \int_0^\infty x^{\alpha} e^{-x} \, dx$$

$u = x^{\alpha}$ $\qquad dv = e^{-x} \, dx$

$du = \alpha x^{\alpha-1} \, dx$ $\qquad v = -e^{-x}$

$$\Gamma(\alpha+1) = -x^{\alpha} e^{-x}\Big|_0^{\infty} + \alpha \underline{\int_0^{\infty} x^{\alpha-1} e^{-x}\, dx}$$

$\rightarrow e^{-\infty} = 0 \;,\; e^{0} = 1$

$$\boxed{\therefore \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)}$$

---

ex:2 show that $\Gamma(n) = (n-1)!$ , $n$ integer

**sol**

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)$$

$$= (n-1)(n-2)(n-3)\Gamma(n-3)$$

$$= (n-1)(n-2)(n-3)\cdots\cdots\Gamma(1)$$

$$\Gamma(1) = \int_0^{\infty} e^{-x}\, dx = -e^{-x}\Big|_0^{\infty} = -\left[e^{-\infty} - e^{0}\right] = 1$$

$$\Gamma(n) = (n-1)!$$

~~1 كيفية حساب Γ( [illegible] )~~

2 Lec 21

$\boxed{1}$ كيفية حساب $\Gamma$ ( رقم )

عدد سالب

* نستخدم

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$$

عدد صحيح موجب

* نطرح منه واحد ونضعه داخل المضروب

$$\Gamma(n) = (n-1)!$$

كسر موجب

* نعوض قيمة " $\Gamma$ " حتى تنحصر بين "1 ، 0" باستخدام

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

Ex: Evaluate

① $\Gamma\left(\frac{5}{2}\right)$ ② $\Gamma\left(\frac{-5}{2}\right)$ ③ $\Gamma(7)$

$\boxed{\text{Sol}}$

① $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2}$

$$= \frac{3}{4}\sqrt{\pi}$$

نزود (1) ونقسم على القديمة

② $\Gamma\left(\frac{-5}{2}\right) = \dfrac{\Gamma\left(\frac{-3}{2}\right)}{\frac{-5}{2}}$

$$= \frac{-2}{5}\left[\frac{\Gamma\left(\frac{-1}{2}\right)}{\frac{-3}{2}}\right] = \frac{4}{15}\left(\frac{\Gamma\left(\frac{1}{2}\right)}{\frac{-1}{2}}\right)$$

$$= \frac{-8}{15}\pi$$

$\boxed{\text{Ex}}$ show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\boxed{\text{Sol}}$

$$\Gamma(n) = \int_0^\infty u^{n-1} e^{-u} \, du$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty u^{\frac{-1}{2}} e^{-u} \, du$$

Put

$u = x^2$

$du = 2x \, dx$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1} e^{-x^2} * 2x \, dx$$

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-x^2} \, dx$$

$$\rightarrow \Gamma\left(\frac{1}{2}\right)^2 = \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)$$

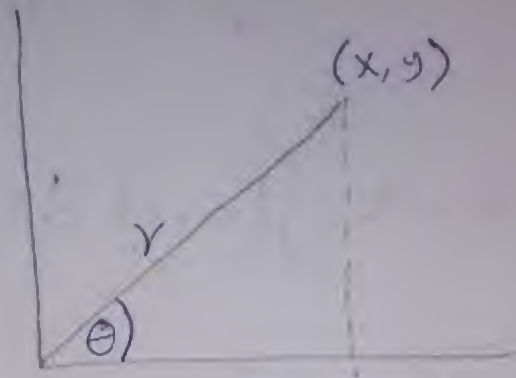
$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \left(2 \int_0^\infty e^{-x^2} \, dx\right)\left(2 \int_0^\infty e^{-y^2} \, dy\right)$$

$$= 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$$

← نستخدم الإحداثيات القطبية

$$x = r\cos\theta$$

$$y = r\sin\theta$$

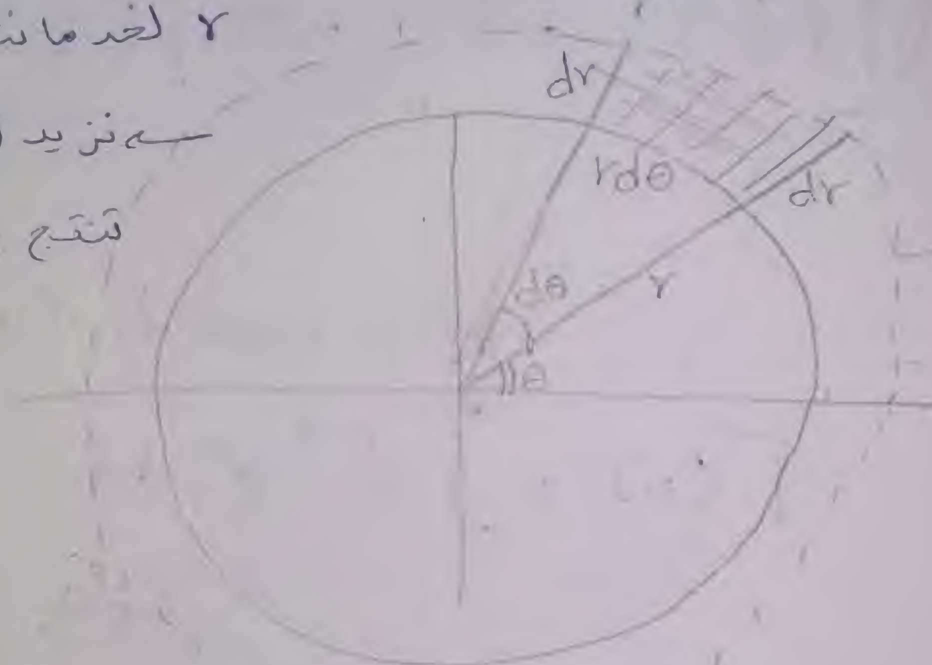


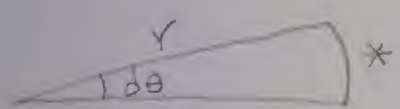

← احنا هنا قمنا بلف المنطقة (الخيط)

r لحد ما نتج دائرة نصف قطرها r

← نزيد الخيط بقيمة (dr) وندوره

تنتج دائرة أكبر.




$$\sin d\theta = \frac{*}{r}$$

$$* = r\sin d\theta = r\,d\theta \qquad 0 < r < \infty \,,\, 0 < \theta < \frac{\pi}{2}$$

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = 4\int_0^{\frac{\pi}{2}}\left(\int_0^{\infty} e^{-r^2}\, r\,dr\right)d\theta$$

$$= \frac{4}{-2}\int_0^{\frac{\pi}{2}}\left(\int_0^{\infty} e^{-r^2}\, -2r\,dr\right)d\theta$$

$$= -2 \int_0^{\frac{\pi}{2}} \left( e^{-r^2} \Big|_0^{\infty} \right) d\theta$$

$$= -2 \int_0^{\frac{\pi}{2}} [0-1]\, d\theta = 2\, \theta \Big|_0^{\frac{\pi}{2}}$$

$$\left(\Gamma\left(\tfrac{1}{2}\right)\right)^2 = \pi \implies \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

---

## $\boxed{2}$ أفكار التكامل

* نحاول توصيل رأس المسألة إلى صورة تكامل

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

### الأفكار

a) اللى فوق e غير جاهز ⟸ نوضع ~~ال~~ ما فوق e بسالب رمز.

(ex) $\int x^2 e^{-5x}\, dx$

b) يظهر داخل التكامل (رقم) $a^{f(x)}$ دالة $\implies$ (ex) $a^{f(x)}$

Let: $a^{f(x)} = e^{\ln a^{f(x)}} = e^{f(x) \ln a}$

$\boxed{6}$ Lec 21

c) يظهر داخل التكامل $\ln x$ والحدود من 0 إلى 1.

نضع $\ln x = -t \Leftarrow x = e^{-t}$

$\ln 0 = -\infty \ , \ \ln 1 = 0$

$\boxed{EX}$ Evaluate

① $\int_0^\infty x^3 e^{-2x} \cosh x \, dx$ ② $\int_3^\infty e^{6x - x^2}$

③ $\int_0^\infty \sqrt{x}\, e^{-x^3} \, dx$ ④ $\int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)} \, dx$

⑤ $\int_0^\infty (1+\sqrt{x})^2 e^{-x} \, dx$ ⑥ $\int_0^\infty x^2 \, 3^{-x} \, dx$

⑥ $\int_{-\infty}^\infty e^{(3x - e^x)} \, dx$ ⑧ $\int_0^{\frac{1}{2}} x^{m-1} \ln\left(\frac{1}{2x}\right) dx$

solution

①
$$I = \int_0^\infty x^3 e^{-2x} \left[\frac{e^x + e^{-x}}{2}\right] dx$$

$$= \underbrace{\frac{1}{2}\int_0^\infty x^3 e^{-x} \, dx}_{I_1} + \underbrace{\frac{1}{2}\int_0^\infty x^3 e^{-3x} \, dx}_{I_2}$$

$$I_1 = \frac{1}{2}\int_0^\infty x^3 e^{-x}\,dx = \frac{1}{2}\Gamma(4) = \frac{1}{2}(3!)$$

$$I_2 = \frac{1}{2}\int_0^\infty x^3 e^{-3x}\,dx$$

$$u = 3x \longrightarrow x = \frac{u}{3} \longrightarrow dx = \frac{du}{3}$$

$$I = \frac{1}{2}\int_0^\infty \frac{u^3}{27}\, e^{-u} * \frac{du}{3}$$

$$= \frac{1}{2\cdot 3^4}\int_0^\infty u^3 e^{-u}\,du = \frac{1}{2\cdot 3^4}\Gamma(4) = \frac{3!}{2 * 3^4}$$

$$I = I_1 + I_2 = \frac{1}{2}(3!) + \frac{3!}{2 * 3^4}$$

---

4 (4)

$$I = \int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)}\,dx = \int_0^1 \sqrt{\ln x^{-1}} = \int_0^1 \sqrt{-\ln x}\,dx$$

$$\ln x = -u \;,\; x = e^{-u} \Rightarrow dx = -e^{-u}\,du$$

$$\text{at } x = 0 \Rightarrow \ln 0 = -u \Rightarrow u = \infty$$

at $x = 1 \Rightarrow \ln 1 = -u \Rightarrow u = 0$

$$I = \int_{\infty}^{0} u^{\frac{1}{2}} e^{-u} \, du = \int_{0}^{\infty} u^{\frac{1}{2}} e^{-u} \, du$$

$$= \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

---

$\boxed{7}$ $$I = \int_{-\infty}^{\infty} e^{(3x - e^{x})} \, dx$$

$$= \int_{-\infty}^{\infty} e^{3x} \cdot e^{-e^{x}} \, dx = \int (e^{x})^{3} e^{-e^{x}} \, dx$$

$u = e^{x}$ $\qquad dx = \dfrac{du}{u}$

$e^{-\infty} = 0 \quad , \quad e^{\infty} = \infty$

$$I = \int_{0}^{\infty} u^{3} e^{-u} \frac{du}{u} = \int_{0}^{\infty} u^{2} e^{-u} \, du$$

$$= \Gamma(3) = 2!$$

$\boxed{9}$ Lec 21